New black holes in the brane world?

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It is known that the Einstein field equations in five dimensions admit more general spherically symmetric black holes on the brane than four-dimensional general relativity. We propose two families of analytic solutions (with $g_{tt} \neq -g_{rr}^{-1}$), parametrized by the Arnowitt-Deser-Misner mass and the parametrized post-Newtonian parameter β, which reduce to Schwarzschild for $\beta = 1$. Agreement with observations requires $|\beta - 1| \sim |\eta| \ll 1$. The sign of η plays a key role in the global causal structure, separating metrics which behave like Schwarzschild ($\eta < 0$) from those similar to Reissner-Nordström spare-time ($\eta > 0$). In the latter case, we find a family of black hole space-times completely regular.

In recent years there has been a renewed interest in models with extra dimensions in which the standard model fields are confined to our four-dimensional world viewed as a (infinitely thin) hypersurface (the brane) embedded in the higher-dimensional space-time (the bulk) where (only) gravity can propagate. Of particular interest are cases where the extra dimensions are infinitely extended but "warped" by the presence of a nonvanishing bulk cosmological constant Λ related to the (singular) vacuum energy density of the brane [1,2] by the standard junction equations [3].

In $D+1$ space-time dimensions a vacuum solution must satisfy ($\mu, \nu = 0, \dots, D$)

$$\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu}. \qquad (1)$$

On projecting the above equation on a time-like manifold of codimension one (the brane) and introducing Gaussian normal coordinates x^i ($i = 0, \dots, D-1$) and z ($z = 0$ on the brane), one obtains the constraints (at $z = 0$)

$$\mathcal{R}_{iz} = 0, \quad R = \lambda, \qquad (2)$$

where R is the D-dimensional Ricci scalar, λ the cosmological constant on the brane (we shall set $\lambda = 0$ from now on, equivalently to the fine tuning between Λ and the brane tension [1]) and use has been made of the necessary junction equations [3]. For static solutions, one can view Eqs. (2) as the analogs of the momentum and Hamiltonian constraints in the Arnowitt-Deser-Misner decomposition of the metric and their role is therefore to select out admissible field configurations along hypersurfaces of constant z. Such field configurations will then be "propagated" off-brane by the remaining Einstein equation (1). It is clear that the above "Hamiltonian" constraint is a weaker requirement than the purely D-dimensional vacuum equations $R_{ij} = 0$ and, in fact, it is equivalent to $R_{ij} = E_{ij}$ where E_{ij} is (proportional to) the (traceless) projection of the $D+1$-dimensional Weyl tensor on the brane [4].

In the present paper we investigate spherically symmetric solutions to Eqs. (2) with $D = 4$ of the form

$$ds^2 = -N(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2 \qquad (3)$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, which might represent black holes in the brane-world [5–9]. First of all, let us recall that the Schwarzschild four-dimensional metric (3) with

$$N = A^{-1} \qquad (4)$$

and $N = 1 - 2M/r$ is ruled out as a physical candidate since its unique propagation in the bulk is a black string with the central singularity extending all along the extra dimension and making the AdS horizon singular [5]. Further, this case is also unstable under linear perturbations [10]. A few different cases have been recently investigated with the condition (4) [6,7]. We stress that while Eq. (4) is accidentally verified in four dimensions, there is no reason for it to hold in this scenario as well. It is easy to show that the most general solution satisfying Eq. (4) is of the "Reissner-Nordström" (RN) type [6]

$$N = 1 - \frac{2M}{r} + \frac{Q}{r^2}, \qquad (5)$$

where Q can be interpreted as a "tidal charge." If instead one insists on requiring the Schwarzschild metric on the brane but with a regular AdS horizon the price to pay is to have matter in the bulk [7].

We shall here present two new families of analytic solutions of the form (3) on the brane (at $z = 0$) obtained by relaxing the condition (4). They are determined by fixing alternatively N or A as in Schwarzschild [so to have the correct $O(1/r)$ behavior at large r] and finding the most general solutions for the constraints (2). These solutions will be expressed in terms of the ADM mass M and the parameterized post-Newtonian (PPN) parameter β which affects the perihelion shift and the Nordtvedt effect [11]. The momentum constraints are identically satisfied by the form (3) and the "Hamiltonian" constraint can be written out explicitly in terms of N and A as

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$$\frac{1}{2}\frac{N''}{N}-\frac{1}{4}\left(\frac{N'}{N}\right)^2-\frac{1}{4}\frac{N'}{N}\frac{A'}{A}-\frac{1}{r}\left(\frac{A'}{A}-\frac{N'}{N}\right)-\frac{1}{r^2}(A-1)=0. \quad (6)$$

Case I. By demanding that

$$N=1-\frac{2M}{r}, \quad (7)$$

the general solution to Eq. (6) is

$$A=\frac{\left(1-\dfrac{3M}{2r}\right)}{\left(1-\dfrac{2M}{r}\right)\left[1-\dfrac{M}{2r}(4\beta-1)\right]}. \quad (8)$$

The only other non-vanishing PPN parameter (which controls the deflection and time delay of light [11]) $\gamma=\beta$ and one knows that $\beta\simeq\gamma\simeq1$ from solar system measurements [11]. In particular the combination $\eta\equiv4\beta-\gamma-3=3(\beta-1)$ measures the difference between the inertial mass and the gravitational mass of a test body. We also note that the solution (8) depends on just one parameter and for $M\rightarrow0$ one recovers the Minkowski vacuum. The same is true for the solution (5) provided one defines $Q=2M^2(\beta-1)$, which yields

$$N=\frac{1}{A}=1-\frac{2M}{r}+\frac{2M^2}{r^2}(\beta-1). \quad (9)$$

As was noted in [9,6], for a (point-like) matter source located on the brane Q should be related to M through the brane tension, therefore measuring β would give information on the vacuum energy of the brane-world (or, equivalently, Λ). Finally, the metric (9) can be experimentally distinguished from the case presented here since the corresponding $\gamma=1$ and does not depend on β.

Case II. Upon setting

$$A^{-1}=1-\frac{2\gamma M}{r}, \quad (10)$$

one obtains

$$N=\frac{1}{\gamma^2}\left(\gamma-1+\sqrt{1-\frac{2\gamma M}{r}}\right)^2, \quad (11)$$

where now $\gamma=2\beta-1$ and this case can be experimentally distinguished both from Eq. (9) and case I above. Finally, one has $\eta=\gamma-1=2(\beta-1)$.

We shall now explore the causal structure of the previous solutions by expressing the metric elements in terms of $\eta\sim\beta-1\neq0$ (keeping in mind that $|\eta|\ll1$ from experimental data [11]). In particular, we shall show that the sign of η is of great relevance.

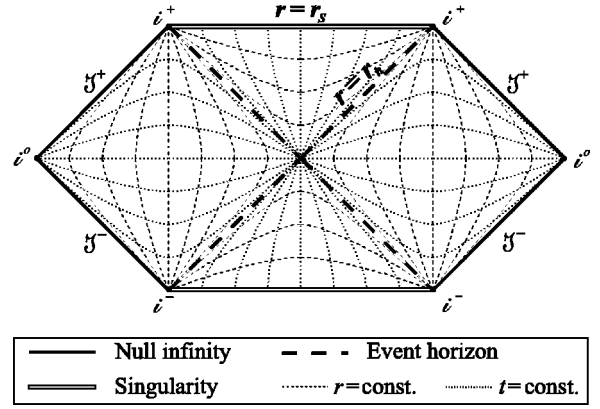*Case I.* The metric components are given by Eq. (7) and

FIG. 1. Penrose diagram for case I and $\eta<0$.

$$A=\frac{\left(1-\dfrac{3M}{2r}\right)}{\left(1-\dfrac{2M}{r}\right)\left[1-\dfrac{3M}{2r}(1+\bar{\eta})\right]}, \quad (12)$$

where we have defined $\bar{\eta}\equiv(4/9)\eta$. As in Schwarzschild, the event horizon is at $r=r_h\equiv2M$ and the related Hawking temperature is

$$T_H=\frac{\sqrt{1-3\bar{\eta}}}{8\pi M}. \quad (13)$$

We see that, with respect to the case $\bar{\eta}=0$, T_H is either slightly reduced or augmented depending on the sign of $\bar{\eta}$. Inside the event horizon, the line element presents singularities in the metric for $\bar{\eta}\neq0$ at

$$r=\begin{cases}\dfrac{3}{2}M(1+\bar{\eta})\equiv r_0,\\[2mm]\dfrac{3}{2}M\equiv r_s.\end{cases} \quad (14)$$

Note that for $\bar{\eta}>0$ it is $r_s<r_0$, whereas for $\bar{\eta}<0$ we have $r_s>r_0$. Calculation of the curvature invariants

$$R^2\equiv R_{\alpha\beta}R^{\alpha\beta}, \quad K^2\equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}, \quad (15)$$

shows the presence of a physical singularity at r_s, where $R^2\sim K^2\sim\bar{\eta}^2/(r-r_s)^4$. Therefore as $\bar{\eta}<0$ the space-time ends at $r=r_s$ and the Penrose diagram, similar to that of Schwarzschild, is represented in Fig. 1.

The case $\bar{\eta}>0$ is, in a sense, more interesting and deserves further investigation. For $r=r_0$ the squared surface gravity,

$$\kappa^2=\frac{M^2}{r^4}\frac{r-r_0}{r-r_s}, \quad (16)$$

vanishes (an analogue surface is present in RN in the region between the two horizons) and the curvature invariants are

regular [they behave as $\sim 1/(\bar{\eta}^2 M^4)$]. Inspection of the equation for the geodesics of energy E and angular momentum L,

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{r-r_0}{r-r_s}\left[E^2 - \left(\frac{L^2}{r^2}-\epsilon\right)\left(1-\frac{r_h}{r}\right)\right], \quad (17)$$

shows that the space-like surface $r=r_0$ is a turning point for all types of curves ($\epsilon=0,\pm 1$, respectively, for null, time-like and space-like geodesics). A similar phenomenon occurs in RN, where all curves with $L\neq 0$ get reflected at some point $r=r_f(E,L)<r_-\equiv M-\sqrt{M^2-Q}$. However, space-like and null curves with $L=0$ are able to reach the RN time-like "repulsive" singularity $r=0$. In our case no curve is able to enter the region $r<r_0$, where the signature of the metric is Euclidean. This makes a substantial difference since the true singularity at $r_s<r_0$ does not belong to the physical space. Furthermore, integration of Eq. (17) for $r\sim r_0$ yields the proper time $\tau\sim\sqrt{(r-r_0)(r_0-r_s)}$ and continuation of the physical trajectories across $r=r_0$ can be achieved, e.g., by introducing the coordinate $x\equiv\sqrt{r-r_0}$ for $r>r_0$ and then going to negative values of x. In the (t,x,θ,ϕ) coordinate frame, the metric is given by

$$ds^2 = -\left(\frac{x^2+r_0-r_h}{x^2+r_0}\right)dt^2 + \frac{4(x^2+r_0)(x^2+r_0-r_s)}{x^2+r_0-r_h}dx^2$$
$$+ (x^2+r_0)^2 d\Omega^2. \quad (18)$$

Since the metric in Eq. (18) is even under $x\rightarrow -x$, both sides of $x=0$ have the same causal structure. Indeed, in the region $x<0$ one can introduce a new radial coordinate r' such that $x=-\sqrt{r'-r_0}$ for which the solution looks exactly like that in Eqs. (7) and (12). The full Penrose diagram is given in Fig. 2. Unlike RN, the space-time is completely regular (the geometry is that of a traversable wormhole with the minimal sphere inside the horizon) and continuation beyond the Cauchy horizon is determined solely by boundary conditions at asymptotically flat regions. Regular four-dimensional black holes were constructed in [12] by matching Schwarzschild with de Sitter along a space-like surface at $r_j<r_h$, and by gluing black hole with white hole metrics in [13] as candidate space-times where information is not lost during the evaporation process. Contrary to those cases, it is important to stress that the continuation of our solution across $x=0$ is smooth, as the extrinsic curvature is continuous in x (and vanishes at $x=0$).

Case II. The metric elements can now be written

$$N = \left[\frac{\eta+\sqrt{1-\frac{2M}{r}(1+\eta)}}{1+\eta}\right]^2$$

$$A = \left[1-\frac{2M}{r}(1+\eta)\right]^{-1}. \quad (19)$$

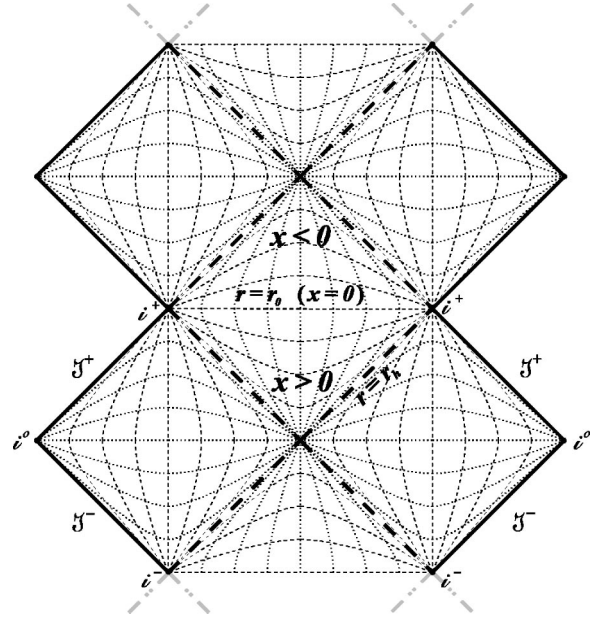We shall again distinguish between the two cases $\eta<0$ and $\eta>0$.

FIG. 2. Penrose diagram for case I and $\eta>0$. The full diagram can be obtained by repeating the same structure infinitely many times both in the future and past.

For $\eta<0$ the metric is singular at

$$r = \begin{cases} \dfrac{2M}{1-\eta}\equiv r_h, \\[2mm] 2M(1+\eta)\equiv r_0, \end{cases} \quad (20)$$

where $r_h>r_0$. r_h defines the event horizon with formal Hawking temperature

$$T_H = \frac{(1-\eta)^2}{8\pi M}. \quad (21)$$

However, the null surface $r=r_h$ is singular since

$$R^2 \sim K^2 \sim \frac{\eta^2}{M^3(\sqrt{r-r_0}-\sqrt{r_h-r_0})^2}. \quad (22)$$

The corresponding Penrose diagram is represented by the left diagram in Fig. 3.
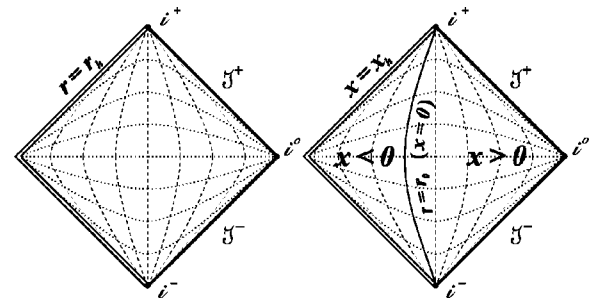

FIG. 3. Penrose diagram for case II with $\eta<0$ (left) and $\eta>0$ (right).

Turning to the case $\eta > 0$, we note that the only singularity in the metric is at $r = r_0$, where all curvature invariants are regular. Similarly to the corresponding case I, the geodesic equation near r_0 is

$$\left(\frac{dr}{d\tau}\right)^2 \sim \frac{r-r_0}{r_0}\left[\frac{(1+\eta)^2}{\eta^2}E^2 - \frac{L^2}{r_0^2} + \epsilon\right], \qquad (23)$$

and $r = r_0$ is a turning point for all physical curves. Smooth continuation at $r = r_0$ is achieved by considering negative values of $x \equiv \sqrt{r - r_0}$. In terms of the radial coordinate x the line element is

$$ds^2 = -\frac{1}{(1+\eta)^2}\left(\eta + \frac{x}{\sqrt{x^2+r_0}}\right)^2 dt^2 + 4(x^2+r_0)dx^2$$
$$+ (x^2+r_0)^2 d\Omega^2. \qquad (24)$$

Unlike the corresponding case I, however, the two sides of $x = 0$ are not symmetric and the metric for negative x exhibits a singular event horizon at $x_h \equiv -\sqrt{r_h - r_0}$ [r_h is given in Eq. (20) and $R^2 \sim K^2 \sim \eta^2/(M^3(x-x_h)^2)$]. The causal structure is described by the right diagram in Fig. 3. The difference with respect to the left diagram is that the physical radius $r = x^2 + r_0$ reaches a minimum at the time-like surface $r = r_0$ ($x = 0$) and then re-expands when x turns to negative values.

An interesting aspect from the above analysis is that there seems to be a correlation between the sign of η (a quantity measured at infinity) and the geometric structure of the solutions at small r (i.e., of the order of the Schwarzschild radius for $|\eta| \ll 1$). For $\eta < 0$, a typical trajectory approaching and possibly entering the horizon is such that the physical radius always decreases (as in Schwarzschild, i.e., $\eta = 0$) and hits the singularity at $r_s > 0$ (larger than $r_s = 0$ for $\eta = 0$). Considering positive values of η, there is always a turning point at $r = r_0$, as an anti-gravity effect occurring very close to r_h in regions not yet experimentally tested. A similar feature is present for the metric (9),

$$N = \frac{1}{A} = 1 - \frac{2M}{r} + \frac{\eta M^2}{2r^2}. \qquad (25)$$

For $\eta < 0$ there is a single horizon at $r_h = M(1 + \sqrt{1 - \eta/2})$ with the Hawking temperature

$$T_H = \frac{1}{2\pi M}\frac{\sqrt{1-\eta/2}}{(1+\sqrt{1-\eta/2})^2}, \qquad (26)$$

and a physical space-like singularity at $r_s = 0$ (the Penrose diagram is the same as that in Fig. 1). If $0 < \eta < 2$ (RN), there are two horizons at $r_\pm = M(1 \pm \sqrt{1-\eta/2})$ [the event horizon is $r_h = r_+$ and the Hawking temperature is still given by Eq. (26)] and the time-like singularity at $r_s = 0$ becomes "repulsive." It would be interesting to inspect whether the sign of η plays the same role in general. From a physical point of view, since anti-gravity effects on the brane are expected for negative brane vacuum energy σ [4], we suspect that (at least for the cases considered here) the sign of η is minus the sign of σ.

We also note that for finite values of η each family of solutions possesses a zero temperature black hole: $\bar{\eta} = 1/3$ for case I, $\eta = 1$ for case II and the well-known extreme RN $\eta = 2$ in Eq. (25). Contrary to the extreme RN which is singular at $r = 0$, the first two solutions are instead completely regular and, although the corresponding values of η are ruled out on atrophysical scales, they might be acceptable candidates as small black holes [8,9]. We will give more details elsewhere.

Let us finally mention that it will be important to investigate the extension of our solutions into the bulk. For Schwarzschild, the singularity at $r = 0$ on the brane extends into the bulk and makes the AdS horizon singular as well. However, for $\eta > 0$ the solutions with (7) and (8) are remarkably free of singularities and one might hope that the bulk is regular as well. This study can be attempted either numerically or by Taylor expanding all five-dimensional metric elements in powers of the extra coordinate. The latter method is currently being investigated.

Note added. After completion of this work we learned that our case I was also derived in Ref. [14] as a possible metric outside a star on the brane. In [14] no attempt was made to study the causal structure of the space-time.

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